Dade's Ordinary Conjecture for the Finite Special Unitary Groups: Part III

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Abstract

Let G be a finite group. An ordinary character of G is the character of a representation of G over a field of characteristic 0. In the p-modular representation theory of G,where p is a prime dividing the order of G, the ordinary irreducible characters of G are divided into disjoint sets called p-blocks which reflect the decomposition of the group algebra of G over a field of characteristic p into indecomposable two-sided ideals. An important problem is to classify the p-blocks, and a first step is to count the number of ordinary characters in a block.

The aim of Dade's Ordinary Conjecture (DOC) is to prove an alternating sum of the form

$$\sum_{C/G} (-1)^{|C|} k(N_G(C), B, d) = 0, \qquad \forall d \geq 0$$

which counts the number of characters in B in terms of corresponding numbers in subgroups of G which are normalizers of chains of certain p-subgroups of G.

This has been shown for p-blocks, p dividing q, for $\mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$ and $\mathrm{U}_n(q)$. Moreover, we have proved DOC for $\mathrm{SU}_n(q)$. The main difficulties involved arose because the structure of the unitary groups is more complicated than that of the linear groups. In particular the cancellations in the alternating sum in the unitary case are very different from the cancellations that occur in the general linear case. A key result utilized is that a version of the parametrization of characters used by Ku for $\mathrm{U}_n(q)$ survives restriction to $\mathrm{SU}_n(q)$.

This report is devoted to presenting an example which aims to elucidate cancellation that occurs in the previously described sub-sums.

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1 Introduction

This introduction to Part III of this three part series includes the main result from Part I and Part II. While some definitions are restated for clarity, the reader is encouraged to revisit Part I. In particular Section 2.3 of Part I will remind the reader of the definition of function β in the statements of the main results below.

1.1 Statement of the Main Theorem

Recall the set up in Section 5 of Part I. Let $J \subset I = [m]$, an index set for the distinguished generators of the Weyl group of type B_m. Let P_J be the standard parabolic subgroup of $\mathrm{U}_n(q)$ corresponding to J. Then $P_J = \bigcap_{j \in J} P_j$ where P_j is the maximal parabolic subgroup corresponding to j. We have the usual Levi decomposition of $P_J = L_J U_J$ where L_J is a levi subgroup and U_J is the unipotent radical of P_J.

Recall that $k_d(P_J, \rho, \det, j)$ denotes the number of irreducible ordinary characters χ of the parabolic subgroup P_J with q-height d and lying over the central character ρ such that the restriction of χ to the kernel of the map ϕ has j' irreducible components, where j divides j'. Moreover the q-height of χ is d if $q^d \| \chi(1)$.

Theorem 1.1 (Main) *Let $Z = Z(\mathrm{U}_n(q))$ and $\{P_J | J \subseteq I\}$ the set of standard parabolic subgroups in $\mathrm{U}_n(q)$. For any $\rho \in \mathrm{Irr}(Z)$, any positive integer j, and all nonnegative integers d we have*

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, \rho, \det, j) = \begin{cases} \beta((n), a_\rho), & \text{if } d = \binom{n}{2} \text{ and } j{=}1; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.1 is a consequence of the following proposition. Distinguish between characters χ counted by $k_d(P_J, \rho, \det, j)$ for which $\ker \chi$ contains U_J or not.

Definition 1.2 *Let $k_d^0(P_J, U_J, \rho, \det, j)$ be the number of characters counted by $k_d(P_J, \rho, \det, j)$ which contain U_J in their kernel and let $k_d^1(P_J, U_J, \rho, \det, j)$ count those characters which do not contain U_J in their kernel.*

Proposition 1.3 *For any $\rho \in \mathrm{Irr}(Z)$, any positive integer j, and all nonnegative integers d*

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^0(P_J, U_J, \rho, \det, j) = \sum_{\substack{\mu \vdash n \\ n'(\mu) = d \\ j | \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho) \tag{1a}$$

$$\sum_{J \subseteq I} (-1)^{|J|} k_d^1(P_J, U_J, \rho, \det, j) = - \sum_{\substack{\mu \vdash n \\ n'(\mu) = d \\ j | \gcd(q+1, \lambda(\mu))}} \beta(\mu, a_\rho). \tag{1b}$$

In Part I we proved Equation 1a. In Part II we proved Equation 1b. This part is devoted to a non-trivial example which elucidates the cancellations which occur. In particular, the author hopes this example serves to motivate the parameterization which is used in Equation 1b.

2 The Example: Dimension 4

Let $K = \overline{F}_q$. Let $\widetilde{G} = \mathrm{GL}_4(K)$. Under the Frobenius $F(a_{i,j}) = M(a_{j,i}^q)^{-1}M^{-1}$ we have $\widetilde{G}^F = U_4(q)$ which we denote by G. The Weyl group, $\widetilde{W}$, of $\widetilde{G}$ is of type A_3, i.e. $\widetilde{W} = S_3$, the symmetric group on three elements. Let $\{1, 2, 3\}$ be an index set for the distinguished generators of $\widetilde{W}$. The Weyl group, W, of G is of type B_2. The F-orbits on the reflections is given by $\{\{1, 3\}, \{2\}\}$. Let $I = \{1, 2\}$ index this set. Let $\widetilde{B}$ be the Borel subgroup of upper triangular matrices in $\widetilde{G}$. Observe that $\widetilde{B}$ is F-stable. Let $B = \widetilde{B}^F$. Then B is upper triangular. In keeping with the notation established in parts I and II we have the following parabolic subgroups,

$$P_\emptyset = G, \qquad P_{\{1\}}, \qquad P_{\{2\}}, \qquad P_{\{1,2\}} = B$$

where each P_J has Levi decomposition $L_J U_J$. Writing the Levi subgroups as block matrices, we have the following L_J:

$$L_\emptyset = G = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix} \right\},$$

$$L_{\{1\}} = \left\{ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \cong \mathrm{GL}_1(q^2) \times \mathrm{U}_2(q),$$

$$L_{\{2\}} = \left\{ \begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\} \cong \mathrm{GL}_2(q^2),$$

$$L_{\{1,2\}} = \left\{ \begin{pmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \cong \mathrm{GL}_1(q^2) \times \mathrm{GL}_1(q^2).$$

Writing the unipotent radicals of the P_J as block matrices, we have the following U_J together with their respective derived series:

$$U_\emptyset = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = 1,$$

$$U_{\{1\}} = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} > \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} > 1,$$

$$U_{\{2\}} = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = Z_{\{2\}} > 1,$$

$$U_{\{1,2\}} = \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} > \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = Z_{\{2\}} > 1.$$

For nonempty J, we enumerate the quotient modules for the P_J and orbit representatives.

$J = \{1\}$:

$$U_{\{1\}}/Z_{\{1\}} \cong \left\{ \begin{pmatrix} 1 & a & b & 0 \\ 0 & 1 & 0 & -a^q \\ 0 & 0 & 1 & -b^q \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b \in F_{q^2} \right\} \cong M_{1,2}(q^2) \text{is a unitary module for } L_{\{1\}}.$$

Let τ_s correspond to a singular chain of rank 1 in unitary space of dimension 2.
Let τ_n correspond to a non-singular chain of rank 1 in unitary space of dimension 2.

$$Z_{\{1\}} \cong \left\{ \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid c + c^q = 0 \right\} \cong M_{1,1}(q) \text{is a central module for } L_{\{1\}}.$$

Let $x_1 = (\epsilon)$ be the unique non-trivial orbit representative.

$J = \{2\}$:

$$Z_{\{2\}} \cong \left\{ \begin{pmatrix} 1 & 0 & a & d_1 \\ 0 & 1 & d_2 & -a^q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, d_i \in F_{q^2} \text{ and } d_i + d_i^q = 0 \right\} \cong M_{2,2}(q)$$

is a central module for $L_{\{2\}}$. Let $x_2 = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$ be an orbit representative for elements of rank 2.

$J = \{1,2\}$:

$$U_{\{1,2\}}/U_{\{2\}} \cong \left\{ \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -a^q \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a \in F_{q^2} \right\} \cong M_{1,1}(q^2) \text{ is a general linear module for } L_{\{1,2\}}.$$

$$Z_{\{2\}} \cong \left\{ \begin{pmatrix} 1 & 0 & a & d_1 \\ 0 & 1 & d_2 & -a^q \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, d_i \in F_{q^2} \text{ and } d_i + d_i^q = 0 \right\} \cong M_{2,2}(q)$$

is a central module for

$$B/Z_{\{2\}} \cong \left\{ \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \cong P^{+2}_{\{1\}}.$$

We enumerate the members of E and F:

$$\begin{aligned} E = \big\{ &(\emptyset, \emptyset, 0), \\ &(\{1,2\}, \{1\}, (12)) \\ &(\{1,2\}, \{1,2\}, (12)) \big\}, \text{ and} \\ F = \big\{ &(\emptyset, \emptyset, 0), \\ &(\{1\}, \{1\}, 1), (\{2\}, \{2\}, 2), (\{1,2\}, \{2\}, 2), \\ &(\{1\}, \{1,2\}, 1) \big\}. \end{aligned}$$

First observation: Take $e_1 = (\{1,2\}, \{1\}, (12))$ so that $P(e_1) = P_{\{1,2\}}/U_{\{2\}} \cong P^{+2}_{\{1\}}$ with $V(e_1) = V(1,2) \cong M_{1,1}(q^2)$. Take nontrivial $\tau_g \in \mathrm{Irr}(V(1,2))$ Then

$$T_{P(e_1)}(\tau) = \mathrm{GL}_1(q^2) \ltimes V(e_1) \cong P(e_2) \ltimes V(e_1).$$

where $e_2 = (\{1,2\}, \{1,2\}, (12))$ since $P(e_2) \cong \mathrm{GL}_1(q^2)$ by definition. Also note $V(e_2) = 1$ by definition. Let $g \in T_B(\tau_g)/U_{\{1,2\}}$. As a block matrix g can be written:

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a^{-q} & 0 \\ 0 & 0 & 0 & a^{-q} \end{pmatrix}$$

Second observation: Take $f_1 = (\{1,\}, \{1\}, 1)$ so that $P(f_1) \cong P_{\{1\}}$ with $U(f_1) = U_{\{1\}}$ Then

$$T_{P_{\{1\}}}(\tau_s)/U_{\{1\}} \cong T_{L_{\{1\}}}(\tau_s) = P^2_{\{1\}} \cong \mathrm{GL}_1(q^2) \ltimes M_{1,1}(q) \cong P(f_2) \ltimes U(f_2)$$

where $f_2 = (\{1\}, \{1,2\}, 1)$. Let $g \in T_{L_{\{1\}}}(\tau_s)$. As a block matrix g can be written:

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a^{-q} & 0 \\ 0 & 0 & 0 & a^{-q} \end{pmatrix}, \text{ where } b + b^q = 0.$$

The elements e_2 and f_2 have opposite parity and hence lead to cancellation in the alternating sum in the statement of DOC, i.e. for $j' = j/\gcd(j, q+1, 2)$

$$\begin{aligned} k_d(B, \tau_g, \rho, \det, j) &= k_{d-d'}(T_B(\tau_g)/V(1,2), \rho, \det, j') = k^0_{d-d(e_2)}(P(e_2), V(e_2), \rho, \phi_{e_2}, j_{e_2}) \\ k_d(P_{\{1\}}, \tau_s, \rho, \det, j) &= k_{d-d''}(T_{P_{\{1\}}}(\tau_s)/U_{\{1\}}, \rho, \det, j') \\ &= k^0_{d-d(f_2)}(P(f_2), V(f_2), \rho, \phi_{f_2}, j_{f_2}) + k^1_{d-d(f_2)}(P(f_2), U(f_2), \rho, \phi_{f_2}, j_{f_2}) \end{aligned}$$

where d' is the power of q in $|T_B(\tau_g)\backslash B|$ and d'' is the power of q in $|T_{P_{\{1\}}}(\tau_s)\backslash P_{\{1\}}|$, i.e. $d' = d'' = 0$ Then

$$k^0_{d-d(e_2)}(P(e_2), V(e_2), \rho, \phi_{e_2}, j_{e_2}) + k^0_{d-d(f_2)}(P(f_2), U(f_2), \rho, \phi_{f_2}, j_{f_2}) = 0.$$

With regard to the splitting of characters upon restriction to the kernel of the determinant map in general, observe that 4 is divisible by 1, 2, and 4. Moreover, assuming that $q+1$ is divisible by 4, we may consider $j = 1$, 2, or 4.

Let $j = 4$. Then $k_d(L_J, \rho, \det, 4) = 0$ for nonempty subsets J in $\{1,2\}$ so

$$\begin{aligned} \sum_{J \subseteq I} (-1)^{|J|} k^0_d(P_J, U_J, \rho, \det, 4) &= k_d(\mathrm{U}_4(q), \rho, \det, 4) \\ &= \begin{cases} \beta((1^4), a_\rho) = 1, & \text{if } d = 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We have $k^1_d(P_J, U_J, \rho, \det, 4) = 0$ except for $J = \{1\}$. Take τ_s, the orbit representative discussed above, and consider

$$T_{P_{\{1\}}}(\tau_s)/U_{\{1\}} \cong P^2_{\{1\}} \cong \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a^{-q} & 0 \\ 0 & 0 & 0 & a^{-q} \end{pmatrix} \mid a, b \in F_{q^2}, b + b^q = 0 \right\}.$$

The determinant map restricted to $P(f_2) = P^2_{\{1\}}$ is $\phi_{f_2} = \det^2$. Moreover, 4_{f_2} is the least positive integer such that 4 divides

$$4 \text{ divides } 4_{f_2} \cdot \frac{q+1}{(q+1)/\gcd(q+1, 2)}.$$

so $4_{f_2} = 2$. By definition $d(f_2) = 0$ and indeed the exponent of q in $|T_{P_{\{1\}}}(\tau_s)\backslash P_{\{1\}}|$ is zero as already mentioned.

Take non trivial $x \in \mathrm{Irr}(Z^2_{\{1\}})$. Then

$$T_{P^2_{\{1\}}}(x) = U_1(q) \ltimes Z^2_{\{1\}} \cong \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b \in F_{q^2}, a^{1+q} = 1, \text{ and } b + b^q = 0 \right\}.$$

The determinant map restricted to $T_{P^2_{\{1\}}}(x) = \det^4$. Let j' be the least positive integer such that

$$2 \text{ divides } j' \cdot \frac{\left|P^2_{\{1\}}\right|}{\left|T_{P^2_{\{1\}}}(x) \cdot \ker(\det^4)\right|},$$

then $j' = 1$. Hence

$$\begin{aligned} \sum_{J \subseteq I} (-1)^{|J|} k^1_d(P_J, U_J, \rho, \det, 4) &= -k^1_{d-d(f_2)}(P(f_2), U(f_2), \rho, \phi_{f_2}, j_{f_2}) \\ &= -k^1_d(P^2_{\{1\}}, Z^2_{\{1\}}, \rho, \det^2, 2) \\ &= -k_d(\mathrm{U}_1(q), \rho, \det^4, 1) \\ &= -\begin{cases} \beta((1^4), a_\rho) = 1, & \text{if } d = 0; \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Let $j = 2$, then $k_d(L_J, \rho, \det, 2) = 0$ for subsets $J = \{1\}$ and $\{1, 2\}$. The calculations are repetitive so we will not present them for all for $d = 0$, 1, 2, and 3, but rather present some of the more interesting calculations. Consider $d = 0$. Our first observation is that this case includes the above, since 2 divides 4.

$$\begin{aligned} \sum_{J \subseteq I} (-1)^{|J|} k^0_d(P_J, U_J, \rho, \det, 4) &= k_0(\mathrm{U}_4(q), \rho, \det, 2) - k_0(\mathrm{GL}_2(q^2), \rho, \det^{1-q}, 2) \\ &= q\beta((1^4), a_\rho) - \overline{\beta}((1^2), a_\rho) \\ &= q - (q - 1) \\ &= (q - 1) + 1 - (q - 1) \\ &= 1. \end{aligned}$$

Notice that the remaining character splits into 4 irreducibles upon restriction to the kernel of the determinant map. We have $k^1_0(P_J, U_J, \rho, \det, 2) = 0$ except for $J = \{1\}$, which is worked out above.

Continue to assume that $j = 2$ and consider $d = 1$. Then $k_1(\mathrm{GL}_2(q^2), \rho, \det^{1-q}, 2) = 0$ since 0 and 2 are the only possible q-heights for $\chi \in \mathrm{Irr}(\mathrm{GL}_2(q^2))$. Moreover $k_1(\mathrm{U}_4(q), \rho, \det, 2) =$

0 since 2 does not divide $\lambda(\mu)$ for $\mu = (2, 1^2)$. Hence,

$$\sum_{J \subseteq I} (-1)^{|J|} k_1^0(P_J, U_J, \rho, \det, 4) = 0$$

which doesn't seem like an interesting calculation. However, examining the other side of the alternating sum is somewhat more interesting since we see cancellation. Take $x_2 \in S^z(f_2)$ and $S^z(f_3)$ as above, then

$$T_{P_{\{2\}}}(x_2)/U_{\{2\}} \cong \mathrm{U}_2(q) \text{ and } T_{P_{\{1,2\}}}(x_2)/U_{\{2\}} \cong P^2_{\{1\}}.$$

The exponent of q in

$$\left|\mathrm{U}_2(q)\backslash \mathrm{GL}_2(q^2)\right| = \left|P^2_{\{1\}} \backslash P^{+2}_{\{1\}}\right| \text{ is } 1.$$

If K is the kernel of the determinant map restricted to the stabilizers of x_2, then

$$2 \text{ divides } \left|T_{P_{\{2\}}}(x_2)K \backslash P_{\{2\}}\right| = \left|T_{P_{\{1,2\}}}(x_2)K \backslash P_{\{1,2\}}\right|.$$

Hence,

$$k_d(P_{\{2\}}, x_2, \rho, \det{}^{1-q}, 2) = k_{d-1}(\mathrm{U}_2(q), \rho, \det{}^2, 1) \text{ and}$$
$$k_d(P_{\{1,2\}}, x_2, \rho, \det{}^{1-q}, 2) = k_{d-1}(P^2_{\{1\}}, \rho, \det{}^2, 1).$$

Thus

$$\begin{aligned}\sum_{J \subseteq I} (-1)^{|J|} k_1^1(P_J, U_J, \rho, \det, 2) &= -k_0(\mathrm{U}_2(q), \rho, \det{}^2, 1) + k_0(P^2_{\{1\}}, \rho, \det{}^2, 1) \\ &= 0\end{aligned}$$

since $\binom{2}{2} = 1 > 0$.

Let $j = 1$. Since 1 divides every integer, this case is trivial. Indeed,

$$\sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, U_J, \rho, \det, 1) = \sum_{J \subseteq I} (-1)^{|J|} k_d(P_J, U_J, \rho)$$

which has already been shown by Ku ([3]).

References

[1] K. A. Bird, *Dade's Ordinary Conjecture for the Finite Special Unitary Groups: Part I*, Technical Report 15-1231, Northeastern Illinois University, 2015.

[2] K. A. Bird, *Dade's Ordinary Conjecture for the Finite Special Unitary Groups: Part II*, Technical Report 16-1229, Northeastern Illinois University, 2016.

[3] C. Ku, *Dade's Ordinary Conjecture for the Finite Unitary Groups in the Defining Characteristic*, PhD thesis, California Institute of Technology, 1999.